The Effective Field Theory of Large-Scale Structure

Project Report
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Abstract

The observable universe is homogeneous and isotropic on large scales, but shows inhomogeneities on scales smaller than 100 Kpc. Cosmological perturbation theory describes the large-scale distribution of dark matter outside the regime of small scales, where it breaks down due to the perturbations growing to order one. Moreover, this technique is linear and ignores higher-order coupling which can lead to backreaction. This project explores an effective field theoretic approach to describing cosmological large-scale structure, modelling dark matter on large scales as an effective fluid with viscosity induced by small scale dynamics. This provides a viable alternative to standard perturbation theory and allows for the incorporation of small-scale feedback on large-scale structure.

1 Background

Modern cosmology is based upon the cosmological principle, which posits that the universe is on large scales homogeneous and isotropic. Spacetime is described by the Friedmann-Robertson-Walker (FRW) metric:

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + sin^2\theta d\phi^2)\right]$$

Here, k can take values of $-1, 0, 1$ based on the overall curvature of the universe, and $a(t)$ is the scale factor that encodes the expansion of the universe.

The evolution of the scale factor depending on the density and pressure of the various components in the universe is determined by the Friedmann Equations

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{1}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\Big(\rho + 3p\Big) \tag{2}$$

For each fluid component, we also have the *equation of state*, which relates the pressure to the density,

$$p = w\rho \tag{3}$$

These three equations can be used to solve for the evolution of the scale factor given the components that make up the universe.

The currently successful theory of cosmology is the *Big Bang* theory. This is parameterised by the Lambda-CDM model of the universe, which includes a cosmological constant and cold dark matter. This model assumes general relativity to be the correct theory of gravitation, and goes on to explain the inhomogeneities in the Cosmic Microwave Background, the formation of large scale structure, and the observed acceleration in the expansion of the universe.

Although the universe appears homogeneous on large scales, locally, we can observe that there are inhomogeneities, owing to the presence of galaxies and galaxy clusters. The theory explaining the presence and dynamics of this is called *Structure*

Formation.

The presence of nonlinear structures greater in size than 100 Mpc on smaller scales in the form of galaxy clusters, superclusters, and voids and filaments can be explained through the principle of *gravitational instability.* Jeans calculated the dynamics of the growth of initial density perturbations in a cloud of gas in 1902, and the same principles can be used to model the dynamics of structure in the universe.

Structure in the universe is assumed to have evolved from small initial perturbations in density, that are assumed to have been created by the primordial inflaton field. Structures develop due to the process of gravitational instability and collapse. Structure formation studies the properties and evolution of these instabilities in an expanding universe. A complete derivation of the equations for the evolution of these involving general relativity is involved and complex, so we shall stick to deriving solutions using Newtonian dynamics, which is a reasonable approximation model on the scales involved.

1.1 Newtonian Perturbation Theory

We consider the universe to be an ideal Newtonian fluid, and we study instabilities in terms of a dimensionless *density contrast* parameter that describes the deviation of the density of the fluid at a point relative to the average density of the fluid.

$$\delta(x) = \frac{\rho(x)}{\bar{\rho}} - 1 \tag{4}$$

Considering the fluid to be characterised by its density, pressure, and velocity field, we can write three equations that govern its motion in the presence of a gravitational field ϕ:

$$\frac{D\rho}{dt} + \rho\nabla_r \cdot \vec{u} = 0 \qquad \text{(Continuity)}$$

$$\frac{D\vec{u}}{dt} = -\frac{\nabla_r P}{\rho} - \nabla_r\phi \qquad \text{(Euler)} \tag{5}$$

$$\nabla_r^2\phi = 4\pi G\rho \qquad \text{(Poisson)} \tag{6}$$

Where ∇_r denotes the gradient operator with respect to r.

Here, D/dt is the *convective derivative*, which corresponds to:

$$\frac{Df}{dt} = \frac{\partial f}{\partial t} + \vec{u} \cdot \nabla f \tag{7}$$

To study the evolution of perturbations, we decompose the quantities governing the above equations into a homogeneous background part, plus a small perturbation, and then expand the equations, dropping terms after the first order of small perturbations. Thus u becomes $u + v$, ϕ becomes $\phi + \delta\phi$ and ρ becomes $\rho + \delta\rho$.

In an expanding universe, physical coordinates are related to comoving coordinates by the scale factor:

$$r(\vec{t}) = a(t) \cdot \vec{x} \tag{8}$$

$$\vec{u}(t) = \frac{\dot{a}}{a}\vec{x} + a(t)\frac{d\vec{x}}{dt} \tag{9}$$

Where the *peculiar velocity field* is defined as:

$$\vec{v} = a(t)\frac{d\vec{x}}{dt} \tag{10}$$

If we write the fluid equations in terms of perturbed quantities and change derivatives with respect to r to derivatives with respect to the comoving coordinates x, we get:

$$\dot{\delta} = \frac{1}{a}\nabla \cdot [(1+\delta)\vec{v}] \tag{11}$$

$$\dot{\vec{v}} + \frac{\dot{a}}{a}\vec{v} + \frac{1}{a}(\vec{v} \cdot \nabla)\vec{v} = -\frac{1}{a}\vec{\nabla}\delta\phi - \frac{\vec{\nabla}\delta p}{a\bar{\rho}(1+\delta)} \tag{12}$$

$$\nabla^2\delta\phi = 4\pi G\bar{\rho}a^2\delta \tag{13}$$

For the purposes of our discussion, we will consider only the evolution of adiabatic initial perturbations in cold dark matter. This is the component which influences structure formation the most. An essential property of cold dark matter is that it is *pressureless*, which we will use in our subsequent derivation. Cold dark matter also has no interaction with baryonic matter, and we can neglect the density perturbations of baryons while considering the perturbed Poisson equation. Baryonic matter forms a small fraction of total matter, but for our case, assuming dark matter to be the major component is a good approximation. Our set of equations then becomes, ignoring second-order terms:

$$\dot{\delta} + \frac{1}{a}\vec{\nabla} \cdot \vec{v} = 0 \tag{14}$$

$$\dot{\vec{v}} + \frac{\dot{a}}{a}\vec{v} = -\frac{1}{a}\vec{\nabla}\delta\phi \tag{15}$$

$$\nabla^2 \delta\phi = 4\pi G\bar{\rho}a^2\delta \tag{16}$$

Eliminating the velocity field term, and combining the equations, we get the dynamical equation for the evolution for dark matter perturbations:

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} = 4\pi G\bar{\rho}\delta \tag{17}$$

It is rather instructive to look at this equation qualitatively. It resembles the equation for a damped harmonic oscillator, with the Hubble Parameter acting as a sort of a damping term. Thus, the expansion of the universe damps the clumping of dark matter. Also, we can see how the time rate of growth of the perturbations is directly proportional to the instantaneous value of the perturbation, showing how dark matter aids its own collapse.

Considering a matter dominated universe, we can solve the Friedmann Equations for background cosmology, and write:

$$a = a_0\left(\frac{t}{t_0}\right)^{\frac{2}{3}} \tag{18}$$

$$H(t) = \frac{2}{3t} \tag{19}$$

$$H^2(t) = \frac{8\pi G\bar{\rho}}{3} = \frac{4}{9t^2} \tag{20}$$

Substituting these expressions in the governing equation for density perturbations, we get a second-order differential equation:

$$\ddot{\delta} + \frac{4}{3t}\dot{\delta} = \frac{2}{3t^2}\delta \tag{21}$$

Assuming we have a simple pair of power-law solutions to this equation, and putting in $\delta = t^\gamma$ into the equation, we get a characteristic quadratic equation with roots $\gamma_1 = 2/3$ and $\gamma_2 = -1$. γ_1 describes the *growing mode* solution and γ_2 describes the *decaying mode.* The decaying mode falls off, and is usually considered to be negligible. The growing mode grows proportional to $t^{2/3}$ and is denoted $D_+(t)$. In a matter dominated universe, this is equivalent to growing proportional to a. Thus, we can write the evolution of perturbations as:

$$\delta(x,t) = f(x)D_+(t) \tag{22}$$

Here, $D_+(t)$ is called the *growth factor*. It encodes the complete time evolution of the perturbation. The factor $f(x)$ encodes the spatial properties of the field of the initial density perturbations. This equation tells us that an initial configuration of overdensities in space grows in time. The next section covers the spatial field of overdensities in more detail.

Thus, in this introductory section, we have seen how from the relations governing a Newtonian fluid, we can derive the equations for the growth of small initial overdensities that are supposed to give rise to structures through the processes of gravitational collapse. The overdensities are seen to collapse under the influence of their own gravity, and then create gravitational potential wells to catch baryonic matter and aid the formation of visible structures. Thus, in a way, dark matter forms the canvas on which the visible universe is painted!

1.2 Relativistic Perturbation Theory

For a full relativistic treatment of cosmological perturbations, we need to look at the Robertson-Walker metric that describes the homogeneous universe, and induce metric perturbations that are sourced by disturbances in the energy density and pressure of matter in the universe.

The perturbed metric is

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} \tag{23}$$

Where $\bar{g}_{\mu\nu}$ is the unperturbed background metric.

The Einstein equation relates the perturbations in the metric (spacetime) to the perturbations in the source (stress-energy tensor).

We perform a coordinate transformation and work in the *conformal* metric in which the spacetime is flat:

$$ds^2 = a^2(\tau)[d\tau^2 - \delta_{ij}dx^i dx^j] \tag{24}$$

By convention, we induce perturbations in all three kinds of quantities: the 00 element of the metric has the perturbation in the energy density, the $0i$ and $i0$ perturbations are separate and vectorial in nature, and the space-space component gets a metric perturbation.

$$ds^2 = a^2(\tau)[(1 + 2A)d\tau^2 - 2B_i dx^i d\tau - (\delta_{ij} + h_{ij})dx^i dx^j] \tag{25}$$

We notice three perturbative quantities: a scalar A, a vector B_i, and a 2-rank tensor h_{ij}. We will further split them into decoupled scalar, vector and tensor modes for reasons that will be explained shortly.

We expand the vector perturbation as the sum of a curl-less and divergence-less part:

$$B_i = \partial_i B + \hat{B}_i \tag{26}$$

Similarly, the tensor perturbation is expanded as the sum of a scalar trace, a vector, and a transverse-traceless tensor:

$$h_{ij} = 2C\delta_{ij} + \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)E + \partial_i\hat{E}_j + \partial_j\hat{E}_i + \hat{E}_{ij} \tag{27}$$

We now have decoupled sectors of perturbations:

- Scalars: A, B, C, E
- Vectors (divergence-less): $\hat{B}_i, \hat{E}_i$
- Tensors (transverse, traceless): $\hat{E}_{ij}$

This is called the **Helmholtz-Hodge Decomposition**, that is done to segregate metric sectors that *only* excite the corresponding sectors in the perturbation of the stress-energy tensor $\delta T_{\mu\nu}$. Up to linear order, the Einstein equations for these three kinds of perturbations do not mix. So, for instance, the scalar perturbations combine to perturb only the energy density, and so on.

1.2.1 Gauge Fixing

Perturbations of the metric thus induced are dependent on the choice of coordinates, or the *gauge*. To express the perturbed metric, one chooses a specific way to slice 4-dimensional spacetime into hypersurfaces of constant time. A different choice of the slicing can lead to different definitions of the perturbations, even working to induce fictitious perturbations that are solely artefacts of the choice of gauge, even though the background is homogeneous.

One solution to this problem is to study the changes in the perturbations due to a change of the gauge, and then isolate combinations that are invariant under gauge transformations. Perturbations can then be expressed in terms of these invariant *Bardeen variables*.

Another way is to simply fix the gauge. Consider a general coordinate transformation,

$$x^\mu \mapsto \tilde{x}^\mu = x^\mu + \chi^\mu \tag{28}$$

Where $\chi^0 = M$ and $\chi^i = \partial^i N + \hat{N}^i$.

The freedom to choose M and N enables us to fix the gauge. For example, in the *Newtonian gauge* where $B = 0$ and $E = 0$, the full metric is

$$ds^2 = a^2(\tau)[(1 + 2\Psi)d\tau^2 - (1 - 2\Phi)\delta_{ij}dx^i dx^j] \tag{29}$$

The right hand side of the Einstein equations, the stress-energy tensor, will be explored in detail in the following sections, as the theory develops.

2 Motivations

Standard cosmological perturbation theory is successful at large scales at linear order. The effective field theory approach provides a way to incorporate the effects of nonlinearities and also uses the potential and the peculiar velocities as the perturbative quantities, with no conditions on the matter overdensity.

In the linear regime, the different modes of matter perturbations evolve independently of each other. Going up to non-linear order, two short wavelength modes can combine to create a long wavelength perturbation. Thus, short distance modes can affect the evolution of long wavelength perturbations. The effective field method seeks to integrate out the short modes, and describe long-wavelength dynamics with source terms arising due to short-length modes.

This effective field theory describes all relevant quantities on a macroscopic scale, while the details of the small-scale physics are averaged away. This is similar to the the Chiral Lagrangian, which offers an effective theory of pion interactions at energies lower than the QCD scale.

Conventional perturbation theory is done in terms of the density contrast, which becomes comparable to unity at small scales, weakening the validity of the perturbative expansion. However, the gravitational potential ϕ and the velocities v remain small at short length scales. Thus, the perturbation theory can be arranged in terms of these quantities, leaving the density contrast unconstrained.

The final theory details the evolution of long-wavelength modes in a Friedmann-Robertson-Walker universe, affected by an effective fluid that arises from the short wavelength perturbations. The properties of the effective fluid are encoded in the effective stress-energy tensor $\tau_{\mu\nu}$, which shall be derived in the following sections.

The effective fluid is parameterised by numbers which can be extracted from small-scale N-body simulations, that are computationally cheap. The behaviour of structure on large scales is then entirely described by the theory, without the need for large simulations.

3 The Effective Stress-Energy

3.1 Newtonian Treatment

Since we are interested in analysing perturbations on very small scales, the universe can be assumed to be flat, and the Hubble flow can be ignored. This exercise is thus amenable to a Newtonian treatment. Let us assume the universe is filled with a pressureless Newtonian fluid. As we assume most of the matter in the universe to be dark, the assumption that this fluid has zero pressure is a good one. As outlined in section 1.1, the dynamics of this kind of fluid are governed by the three equations: the **continuity** equation, which is essentially the conservation of mass, the **Euler** equation, which is the conservation of momentum, and the **Poisson** equation, which relates the density of the fluid to the gravitational potential.

$$\dot{\rho}_m + \nabla \cdot (\rho_m \vec{v}) = 0 \tag{30}$$

$$\dot{\vec{v}} + (\vec{v} \cdot \nabla)\vec{v} = -\nabla\Phi \tag{31}$$

$$\nabla^2\Phi = 4\pi G \rho_m \tag{32}$$

First, we naively try to construct a symmetric stress-energy tensor $\tau^{\mu\nu}$ for this fluid, that will remain conserved under the above equations. We know the expressions for the classical energy and momentum of the fluid:

$$E = \int d^3x \big[\rho_m + \frac{1}{2}\rho_m v^2 + \frac{1}{2}\rho_m \Phi\big] \equiv \int d^3x \tau^{00} \tag{33}$$

$$P^i = \int d^3x \rho_m v^i \equiv \int d^3x \tau^{0i} \tag{34}$$

which give us expressions we can use for τ^{00} and τ^{0i}.

We will now impose the conservation of stress-energy on this object. First, going with the i-component of the conservation, we have

$$\partial_\alpha \tau^{\alpha i} = 0 \tag{35}$$
$$= \dot{\rho}_m v^i + \rho_m \dot{v}^i + \partial_j \tau^{ji} \tag{36}$$
$$\tag{37}$$

Using the continuity and Euler equations, this becomes

$$-\partial_j(\rho_m v^i v^j) - \rho_m \partial_i \Phi + \partial_j \tau^{ji} = 0 \tag{38}$$

Given this relation, if we can write $\rho_m \partial_i \Phi$ as a divergence, we can isolate an expression for τ^{ij}. For this, we use Poisson's equation to substitute the expression for ρ_m in the second term, which becomes

$$\rho_m \partial_i \Phi = \frac{1}{4\pi G} \nabla^2 \Phi \partial_i \Phi = \frac{1}{4\pi G} \partial_j \left[\partial_i \Phi \partial_j \Phi - \frac{1}{2} \delta_{ij} (\nabla \Phi)^2 \right] \tag{39}$$

Using this is equation 38, we can easily write the expression for the stress-energy tensor:

$$\tau^{ij} = \rho_m v^i v^j + \frac{1}{8\pi G} \left[2\partial_i \Phi \partial_j \Phi - \delta_{ij} (\nabla \Phi)^2 \right] \tag{40}$$

In the next section, the expressions for the stress-energy tensor are derived more formally using a full relativistic treatment that accounts for second-order nonlinearities and shows more clearly how nonlinear terms gives rise to the effective stress-energy.

3.2 Relativistic Calculation

3.2.1 Metric Perturbations

To begin the derivation of the effective stress-energy, we decompose the full Einstein equation, including non-linear contributions, into three parts:

$$\bar{G}_{\mu\nu} + (G_{\mu\nu})^L + (G_{\mu\nu})^{NL} = 8\pi G T_{\mu\nu} \tag{41}$$

The equations for the background quantities and the linear parts are straightforward. Now, the non-linear part is redefined in terms of the linear part:

$$(G_{\mu\nu})^L = 8\pi G(\tau_{\mu\nu} - T_{\mu\nu}) \tag{42}$$

Where the *effective stress energy tensor* $\tau_{\mu\nu}$ is defined as

$$\tau_{\mu\nu} = T_{\mu\nu} - (G_{\mu\nu})^{NL}/8\pi G \tag{43}$$

To start, we consider the perturbed FRW metric in the Poisson gauge

$$ds^2 = a^2(\tau)\Big[- e^{2\psi}d\tau^2 + 2\omega_i dx^i d\tau + (e^{-2\phi}\delta_{ij} + \chi_{ij})dx^i dx^j\Big] \tag{44}$$

Where ω is a divergence-free vector, and χ is a transverse, traceless tensor.

For the purposes of cosmology, the first order vector and tensor perturbations are ignored, and all perturbations are broken into first and second order portions. Let us express the left-hand side of the Einstein equation for this metric first, before moving on to the details of the perturbation theory itself.

In this report, $\mathcal{H}$ is the conformal Hubble parameter, defined as

$$\mathcal{H} = \frac{1}{a}\frac{da}{d\tau} \tag{45}$$

The Christoffel symbols are:

$$\begin{aligned}
\Gamma^0_{00} &= \mathcal{H} + \dot{\psi} &(46)\\
\Gamma^0_{0i} &= \psi_{,i} + \mathcal{H}\omega_i &(47)\\
\Gamma^i_{00} &= e^{2\psi+2\phi}\psi_{,i} + \dot{\omega}_{,i} + \mathcal{H}\omega_i &(48)\\
\Gamma^0_{ij} &= e^{-2\psi-2\phi}(\mathcal{H} - \dot{\phi})\delta_{ij} + \frac{1}{2}\dot{\chi}_{ij} + \mathcal{H}\chi_{ij} - \frac{1}{2}(\omega i, j + \omega_{j,i}) &(49)\\
\Gamma^i_{0j} &= (\mathcal{H} - \dot{\phi})\delta_{ij} + \frac{1}{2}\dot{\chi}_{ij} - \frac{1}{2}(\omega i, j - \omega_{j,i}) &(50)\\
\Gamma^i_{jk} &= -\phi_{,k}\delta^i_j - \phi_{,j}\delta^i_k + \phi^{,i}\delta_{jk} - \mathcal{H}\omega_i\delta_{jk} + \frac{1}{2}\Big[\chi_{ij,k} + \chi_{ik,j} - \chi_{jk,i}\Big] &(51)
\end{aligned}$$

From these, we compute the Einstein tensor up to second order:

$$G^0_0 = -\frac{e^{-2\psi}}{a^2}\left[3\mathcal{H}^2 - 6\mathcal{H}\dot{\phi} + 3\dot{\phi}^2 - e^{2\psi+2\phi}(\phi^2_{,i} - 2\phi_{,kk})\right] \tag{52}$$

$$G^i_0 = 2\frac{e^{2\phi}}{a^2}\left[\dot{\phi}_{,i} + (\mathcal{H} - \dot{\phi})\psi_{,i}\right] - \frac{1}{2a^2}\omega^i_{,kk} - 2(\mathcal{H}^2 - \dot{\mathcal{H}})\frac{\omega^i}{a^2} \tag{53}$$

$$\begin{aligned} G^i_j = \frac{1}{a^2}\Bigg[& e^{-2\psi}\Big(- (\mathcal{H}^2 + 2\dot{\mathcal{H}}) - 2\dot{\phi}\dot{\psi} - 3\dot{\phi}^2 + 2\mathcal{H}(2\dot{\phi} + \dot{\psi}) + 2\ddot{\phi}\Big) \\ & + e^{2\phi}(\psi_{,k}\psi_{,k} + \psi_{,kk} - \phi_{,kk})\Bigg] \\ & + \frac{e^{2\phi}}{a^2}(\phi_{,ij} - \psi_{,ij} + \phi_{,i}\phi_{,j} - \psi_{,i}\psi_{,j} - \psi_{,i}\phi_{,j} - \phi_{,i}\psi_{,j}) \\ & - \frac{1}{2a^2}\left[(\dot{\omega}^i_{,j} + \dot{\omega}^j_{,i}) + 2\mathcal{H}(\omega^i_{,j} + \omega^j_{,i})\right] + \frac{1}{2a^2}\left[\ddot{\chi}^i_j + 2\mathcal{H}\dot{\chi}^i_j - \chi^i_{j,kk}\right] \end{aligned} \tag{54}$$

Now, as in our case there is no anisotropic stress (at this level), so $\phi = \psi$.

Perturbations in this spacetime are generated by matter perturbations, in the density $\delta\rho$, in the pressure δp, and the peculiar velocities v^i. Standard perturbation theory fails when the magnitude of the density contrast $\delta\rho$ becomes of the order of the background matter density. However, the magnitudes of the potential ϕ and the velocities v^i remain small even when the density contrast grows to order one. This makes these quantities efficient as the variables of a perturbation theory. The density contrast can grow to unrestricted values, with only the magnitudes of potential and velocity curtailed to much less than unity. One important result of this change, is that v^2 terms are now first order quantities. Notice that v^2 is of the same order as $\delta\phi$ (because they are of the order of energy). Now, if δ is a zeroth order quantity, then $\delta\phi$ remains first order, and so does v^2.

Also, the Poisson equation for the perturbed potential and density contrast in an expanding universe is

$$\nabla^2\phi = \frac{3}{2}\mathcal{H}^2\delta \tag{55}$$

From here, we can establish a relation between the order of magnitude of the following four kinds of terms

$$\frac{(\nabla\phi)^2}{\mathcal{H}^2} \sim \frac{\phi\nabla^2\phi}{\mathcal{H}^2} \sim \delta\phi \sim v^2 \tag{56}$$

The scheme of expansion in matter perturbations and metric perturbations is also dependent on the gauge. In our current scenario, the gauge corresponds to density fluctuations being large, and metric perturbations being small. Were we to pick a gauge where the density perturbations vanish, the metric perturbations would be large, disabling the perturbative expansion. In principle, such an expansion scheme is valid for all gauges where metric perturbations are very small compared to matter perturbations.

The objective of rearranging the Einstein equation thus is to express the non-linear part of the Einstein tensor as a linear equation. Now, from the expressions for the Einstein tensor (52,53,54), we can isolate the parts order-by-order.

The background terms are

$$\begin{aligned} -a^2\bar{G}^0_0 &= 3\mathcal{H}^2 && (57)\\ \bar{G}^i_0 &= 0 && (58)\\ -a^2\bar{G}^i_j &= \mathcal{H}^2 + 2\dot{\mathcal{H}} && (59) \end{aligned}$$

The linear part of the Einstein tensor is

$$-\frac{a^2}{2}(G^0_0)^L = \nabla^2\phi - 3\mathcal{H}(\dot{\phi}+\psi) \tag{60}$$

$$\frac{a^2}{2}(G^i_0)^L = [\dot{\phi}+\mathcal{H}\psi]_{,i} + \frac{1}{4}\nabla^2\omega^i + (\mathcal{H}^2 - \dot{\mathcal{H}})\omega^i \tag{61}$$

$$\begin{aligned} \frac{a^2}{2}(G^i_j)^L = {} & \Big[(\mathcal{H}^2 + 2\dot{\mathcal{H}}\psi + \mathcal{H}(2\dot{\phi} + \dot{\psi} + \ddot{\phi} - \frac{1}{2}\nabla^2(\phi-\psi)\Big]\delta^i_j + \frac{1}{2}[\phi-\psi]_{,ij} \\ & - \frac{1}{4}\Big[(\dot{\omega}^i_{,j} + \dot{\omega}^j_{,i}) + 2\mathcal{H}(\omega^i_{,j} + \omega^j_{,i})\Big] + \frac{1}{4}\Big[\ddot{\chi}^i_j + 2\mathcal{H}\dot{\chi}^i_j - \chi^i_{j,kk}\Big] \qquad (62) \end{aligned}$$

And the nonlinear (second-order) parts are

$$-a^2(G^0_0)^{NL} = 12\mathcal{H}^2\psi^2 + 12\mathcal{H}\dot{\phi}\psi + 3\dot{\phi}^2 - \phi_{,k}\phi_{,k} + 4\phi\phi_{,kk} \tag{63}$$

$$\frac{a^2}{2}(G^i_0)^{NL} = 2\phi[\dot{\phi} + \mathcal{H}\psi]_{,i} - \dot{\phi}\psi_{,i} \tag{64}$$

$$\begin{aligned} a^2(G^i_j)^{NL} = \Big[& -4(\mathcal{H}^2 + 2\dot{\mathcal{H}})\psi^2 - 2\dot{\psi}\dot{\phi} - 3\dot{\phi}^2 - 4\mathcal{H}(2\dot{\phi} + \dot{\psi})\psi - 4\psi\ddot{\phi} \\ & + \psi_{,k}\psi_{,k} - 2\phi[\phi - \psi]_{,kk}\Big]\delta^i_j \\ & + 2\phi[\phi - \psi]_{,ij} + \phi_{,i}\phi_{,j} - \psi_{,i}\psi_{,j} - \psi_{,i}\phi_{,j} - \phi_{,i}\psi_{,j} \end{aligned} \tag{65}$$

Remember, with no anisotropic stress, we have $\phi = \psi$.

3.2.2 Matter Perturbations

The definition of the effective stress-energy also includes contributions from the matter stress-energy tensor.

We first introduce a timelike velocity 4-vector for an observer comoving with the fluid:

$$u^\mu = \frac{dx^\mu}{d\tau} \tag{66}$$

Where τ is the proper time in the fluid's rest frame.

Now, for an ideal pressureless fluid like dark matter, the stress energy tensor takes the form

$$T_{\mu\nu} = \rho u_\mu u_\nu \tag{67}$$

We have organised our perturbation theory in terms of the three-velocities of the fluid, v. In order to relate these to the 4-velocity in the fluid's rest frame, we use

$$ds^2 = d\tau^2 = g_{\mu\nu}dx^\mu dx^\nu = g_{\mu\nu}v^\mu v^\nu dt^2 \tag{68}$$

Hence, $\frac{d\tau}{dt}$ can be found, which allows us to relate the three-velocities to the four-velocity. We find the following relations:

$$u^0 = a^{-1}e^{-\psi}\gamma(v) \tag{69}$$

$$u^i = a^{-1}e^{\phi}v^i \tag{70}$$

Where γ denotes the usual Lorentz factor

$$\gamma(v) = \frac{1}{\sqrt{1-v^2}} \approx 1 + \frac{v^2}{2} \tag{71}$$

Armed thus, we can now reformulate the expressions for the stress-energy tensor in terms of the local three-velocities.

$$T_0^0 = -\gamma^2\rho = -\rho(1+v^2) \tag{72}$$

$$T_0^i = -e^{\phi+\psi}\rho v^i \tag{73}$$

$$T_j^i = \rho v^i v_j \tag{74}$$

3.2.3 The Relativistic Effective Stress-Energy

Having obtained the expressions for $(G_{\mu\nu})^{NL}$ already, we can now construct the effective stress-energy tensor through its definition (43).

$$\tau_0^0 = -\rho(1+v^k v_k) - \frac{\phi_{,k}\phi_{,k} - 4\phi\phi_{,kk}}{8\pi G a^2} \tag{75}$$

$$\tau_j^i = \rho v^k v_k - \frac{\phi_{,k}\phi_{,k}\delta_j^i - 2\phi_{,i}\phi_{,j}}{8\pi G a^2} \tag{76}$$

Where $\psi = \phi$.

4 Smoothing the equations

The ultimate aim of this exercise is to construct a theory for the long-wavelength perturbations of matter. This will be accomplished in this section, by filtering out the short-wavelength modes using a window with a characteristic length scale Λ. This process is equivalent to smoothing the equations of motion over domains of size Λ^{-1}, which retains the long-wavelength parts of any variable.

The smoothing of any quantity will be done by taking a convolution product of it with a Gaussian filter of width Λ, W_Λ. If Q denotes a perturbation variable, then the long-wavelength part of it is defined as

$$Q_l = [Q]_\Lambda = \int W_\Lambda(\mathbf{x}-\mathbf{y})Q(\mathbf{y})d^3y \tag{77}$$

The total quantity is

$$Q = Q_l + Q_s \tag{78}$$

Using this method, we now look at the linearised Einstein equation (42).

$$\begin{aligned} \int W_\Lambda(\mathbf{x}-\mathbf{y})G^L_{\mu\nu}(\mathbf{y})d^3y = & \int W_\Lambda(\mathbf{x}-\mathbf{y})(\tau_{\mu\nu}(\mathbf{y}) - \bar{T}_{\mu\nu})d^3y & (79) \\ = & [\tau_{\mu\nu}]_\Lambda - \bar{T}_{\mu\nu} & (80) \end{aligned}$$

The left side is linear in fluctuations. On the right side, the quantity $[\tau_{\mu\nu}]_\Lambda$ can be shown to have the general form

$$[\tau_{\mu\nu}]_\Lambda = [\tau_{\mu\nu}]_l + [\tau_{\mu\nu}]_s + [\tau_{\mu\nu}]_{\partial^2} \tag{81}$$

Here, the portion with the l subscript depends only on the long wavelength quantities, the one with s depends only on the short wavelength quantities, and the third one contains higher order derivative terms, which are usually small and can be neglected. A full proof of this segmentation, and the expressions for the portion of the effective stress-energy tensor influenced only by the short modes is given on page 21 of [1].

Now, we can move $[\tau_{\mu\nu}]_l$ to the left side of the equation 79, so that the right side contains *only* functions of short modes. Effectively, the long-wavelength dynamics are shown to be sourced by short-wavelength terms.

The equations relate the density and velocities of a fluid in terms of long-wavelength smoothed quantities, with only short-wavelength sources. **Long modes evolve in the presence of an effective fluid constructed by the short modes**.

Since we will be satisfied with the Newtonian treatment of the fluid variables, we must look at the smoothed Newtonian fluid equations, which correspond to the equations 30. Appendix A of [1] gives a detailed derivation of this form. The continuity equation is linear in the perturbative quantities and simple to smooth. We will state it later. Here, we simply state the result of smoothing the Euler equation here, which is the most important, as it includes contributions from te effective stress-energy.

$$\rho_l \left[\dot{v}_l^i + v_l^j \nabla_j v_l^i\right] \rho_l \nabla_i \Phi_l = -\nabla_j [\tau_i^j]_s \tag{82}$$

Where

$$[\tau^{ij}]_s = [\rho_m v_s^i v_s^j]_\Lambda + \frac{1}{8\pi G}\left[2\partial_i \Phi_s \partial_j \Phi_s - \delta_{ij}(\nabla\Phi_s)^2\right]_\Lambda \tag{83}$$

These derivations are done for a flat universe, but this can easily be extended to working in comoving coordinates in an expanding universe by adding factors of a and $\dot{a}$ to space and time derivatives. This shall be used in section 5, where the continuity, Euler and Poisson equations will be restated and smoothed similarly for the case of comoving coordinates.

Aside: The Effective Fluid

Equations 75 and 76 describe the content of the effective fluid that is generated by small-scale fluctuations. This acts as an effective background in which long-wavelength modes evolve. To extricate the physical meaning of this effective fluid, we can visualise it as a fluid with some effective density, pressure and anisotropic stress. To this end, we define the quantities

$$\rho_{\text{eff}} = \langle \tau_0^0 \rangle, \qquad 3p_{\text{eff}} = \langle \tau_i^i \rangle, \qquad \Sigma_{\text{eff}} = \langle \hat{\tau}_j^i \rangle \tag{84}$$

Where $\hat{\tau}_j^i$ denotes the traceless part of the tensor.

To ease the expression of these definitions, we define symbols for two correlation functions of the quantities of interest.

$$\kappa_{ij} = \frac{1}{2}\langle (1+\delta)v_i v_j \rangle \tag{85}$$

$$\omega_{ij} = \frac{\langle \phi_{,i}\phi_{,j} \rangle}{8\pi G a^2 \bar{\rho}} \approx \frac{\langle \phi\phi_{,ij} \rangle}{8\pi G a^2 \bar{\rho}} \tag{86}$$

and their traces

$$\kappa = \kappa_{ii} = \frac{1}{2}\langle (1+\delta)v^2 \rangle \tag{87}$$

$$\omega = \omega_{ii} = \frac{\langle \delta\phi \rangle}{2} \tag{88}$$

These definitions can be seen as the kinetic and potential energies of the fluid at the scale corresponding to Λ.

Using these definitions and the expressions 75 and 76, we can easily write down

$$\rho_{\text{eff}} = \langle \tau_0^0 \rangle = -\bar{\rho}(1 + 2\kappa - 5\omega) \tag{89}$$

$$\langle \tau_j^i \rangle = -\bar{\rho}(2\kappa_{ij} + w\delta_{ij} - 2\omega_{ij}) \tag{90}$$

From 90, we can clearly see that the effective pressure $3p_{\text{eff}} = \langle \tau_i^i \rangle = -\bar{\rho}(2\kappa + \omega)$. From this, we can see an enlightening and important result: *virialised scales act like a pressureless fluid*. Virialized objects are characterized by the the virial relation between the kinetic and potential energies

$$2K + W = 0 \tag{91}$$

which is the same as the effective pressure being zero, as shown in the expression 90.

5 The Effective Theory

To construct the equations of motion, we can go back to the fluid equations and write them down *only* for the long wavelength quantities. We drop the subscript l for convenience. Since nonlinear effects are expected to be relevant only at short scales, we can use the Newtonian approximation of the full general relativistic equations while looking at the evolution.

To do this, we have to go back and first look at the continuity (11) and Euler (12) equations for the dark matter fluid *in an expanding universe*. We must apply the smoothing process described in section 4 to these. The continuity equation smoothing is simple and yields

$$\dot{\rho} + 3H\rho + \frac{\partial_i(\rho v^i)}{a} = 0 \tag{92}$$

The Euler equation smoothing is done as in equation 82, and yields

$$\dot{v}^i + Hv^i + \frac{v^j \partial_j v^i}{a} + \frac{\partial_i \phi}{a} = -\frac{1}{a\rho} = \partial_j [\tau_{ij}]_s \tag{93}$$

The source term $[\tau_{ij}]_s$ now includes short wavelength modes and the higher derivative terms $[\tau_{ij}]_{\partial^2}$ after the smoothing procedure is applied on the effective stress energy tensor and the result is broken as in 81. Now, since in our theory, the short modes are not observed explicitly, an average over those will be taken. This means that the source term now depends in some unknown way on the *long* modes and their derivatives only, but we are ignorant about the exact nature of this functional dependence. To get around this, we must remember that the long wavelength perturbations are all small in magnitude, and thus we can expand the source term in terms of the long wavelength quantities and their derivatives.

The expansion of the source term (averaged over the short modes) in terms of the long wavelength density contrast and velocity is, upto first order

$$\langle [\tau_{ij}]_s \rangle = \bar{\rho}\Big[c_s^2 \delta \delta^{ij} - \frac{c_{bv}^2}{Ha}\delta^{ij}\partial_k v^k - \frac{3}{4}\frac{c_{sv}^2}{Ha}\Big(\partial_j v^i + \partial_i v^j - \frac{2}{3}\delta^{ij}\partial_k v^k\Big) + ...\Big] \tag{94}$$

Here, the various velocity terms of the form c_i^2 are parameters that characterise the effective field theory. They can be extracted from relatively small-scale N-body simulations, and [2] defines clearly a set of two-point correlation functions that can be computed from an N-body simulation output, that are related to these parameters.

This effective stress-energy that arises from non-linear fluctuations is equivalent to an anistropic stress. The velocity terms signify the following, physically:

- c_s^2 represents the pressure induced by the small scales, where c_s^2 is the commonly known *sound speed* in cosmology, such that any pressure perturbations $\delta p = c_s^2 \delta$.
- c_{sv}^2 represents the effect of the shear viscosity of the fluid, such that the shear $\eta = 3\bar{\rho}c_{sv}^2/4H$.
- c_{bv}^2 represents the effect of the bulk viscosity of the fluid, such that the bulk viscosity $\zeta = \bar{\rho}c_{bv}^2/H$.

Substituting this expression into 93 and expressing $\rho = \bar{\rho}(1+\delta)$, we get the following set of equations:

$$\dot{\delta} = -\frac{1}{a}\partial_i\Big((1+\delta)v^i\Big) \tag{95}$$

$$\dot{v^i} + Hv^i + \frac{1}{a}v^j\partial_j v^i + \frac{1}{a}\partial_i \phi = -\frac{1}{a}c_s^2 \partial^i \delta + \frac{4c_{bv}^2 + c_{sv}^2}{4Ha^2}\partial^i\partial_j v^j + \frac{3}{4}\frac{c_{sv}^2}{Ha^2} \tag{96}$$

This is supported by the Poisson equation relating the perturbed density and potential

$$\nabla^2 \phi = \frac{3}{2}a^2 H^2 \delta \tag{97}$$

To solve these equations, we shall move to Fourier space, since our ultimate objective is to calculate the matter power spectrum in Fourier space. The organisation of the solutions will be such that the corrections to the power spectrum will be expressed in terms of the power spectrum of the first order matter perturbations.

We will use the velocity divergence $\theta = \partial_i v^i$ to write down our equations. Using that, in Fourier space, the equations 95 and (the derivative of) 96 become

$$a\mathcal{H}\delta' + \theta = -\int \frac{d^3q}{(2\pi)^3}\alpha(\vec{q},\vec{k}-\vec{q})\delta(\vec{k}-\vec{q})\theta(\vec{q}) \tag{98}$$

$$a\mathcal{H}\theta' + \mathcal{H}\theta + \frac{3}{2}\mathcal{H}^2\delta - c_s^2k^2\delta + \frac{c_v^2k^2}{\mathcal{H}}\theta = -\int \frac{d^3q}{(2\pi)^3}\beta(\vec{q},\vec{k}-\vec{q})\theta(\vec{k}-\vec{q})\theta(\vec{q}) \tag{99}$$

Where $\mathcal{H} = aH$ is the conformal Hubble parameter, and the prime denotes $\partial/\partial a$.

The Poisson equation has been used to replace a double derivative of the density contrast. The right hand expressions come from the conversion of the equations to Fourier space, and are given by

$$\alpha(\vec{k},\vec{q}) = \frac{(\vec{k}+\vec{q})\cdot\vec{k}}{k^2} \quad , \quad \beta(\vec{k},\vec{q}) = \frac{(\vec{k}+\vec{q})^2\vec{k}\cdot\vec{q}}{2q^2k^2} \tag{100}$$

Our strategy will be to first ignore all source terms of the right hand sides of these equations, and also ignore all terms that come from the effective stress energy tensor, since they involve second derivatives (a factor of k^2) and thus become effectively third order terms. Ignoring all these, we have first order equations. We will use the two equations to eliminate θ and obtain a second order equation in the variable δ.

Now, we seek to find the Green's function for this equation. This will allow us to reconstruct the full solutions from the source terms we earlier ignored. If the Green's function is $G(a,b)$, then the combined second order equation for the Green's function with a delta-function source term becomes

$$-a^2\mathcal{H}^2(a)\partial_a^2G(a,b) - a(2\mathcal{H}^2(a) + a\mathcal{H}(a)\mathcal{H}'(a))\partial_aG(a,b) + \frac{3}{2}\mathcal{H}^2G(a,b) = \delta(a-b) \tag{101}$$

This equation is to be solved by assuming the homogeneous equation, finding solutions to that, and then patching the solutions using the conditions at the boundary $a = b$

$$G(a,b)|_{a=b} = 0 \qquad , \qquad \frac{\partial}{\partial a}G(a,b)|_{a=b} = 1/(b^2\mathcal{H}^2(b)) \tag{102}$$

Moreover, for the we seek to find the retarded Green's function, so $G(a,b) = 0$ for $a < b$.

For a simple Einstein-de Sitter universe with only dark matter, we can obtain solutions analytically, with the Green's function being a sum of solutions proportional to a^{-1} and a^4, but it is advisable to solve for this Green's function numerically. Once it is known, the corrections to the perturbative solutions can be written down.

Now, to write down the full solutions, we will write the solutions for the homogeneous equation for the first order density perturbations as

$$\delta(k,a) = \frac{D(a)}{D(a_0)}\delta s(k) \tag{103}$$

Where a_0 refers to the scale factor at the current time and is normalised as $a_0 = 1$, and $\delta s(k)$ is the first order density perturbation at current time.

We define the power spectrum $P_{11}(k)$ of first order density perturbations at current time as:

$$\langle \delta s(\vec{k})\delta s(\vec{q})\rangle = (2\pi)^3\delta^{(3)}(\vec{k}+\vec{q})P_{11}(k) \tag{104}$$

Now, from this the *smoothed* power spectrum at current time is easy to write in Fourier space: it picks up one factor of the smoothing function for each δs.

$$P_{11,l}(k) = W_\Lambda^2(k)P_{11}(k) \tag{105}$$

Now, to facilitate a perturbation theory-like solution to these equations and restate the solution in terms of thee two-point correlation function (or the power spectrum) as outlined above, we need to take the two-point correlations of the full perturbative expansion. We write δ as a power series in the small parameter ϵ,

$$\delta = \sum_{i=1} \epsilon^i \delta^{(i)} \tag{106}$$

Then the two-point correlation function becomes, schematically

$$\langle\delta\delta\rangle = \epsilon^2\langle\delta^{(1)}\delta^{(1)}\rangle + \epsilon^4[\langle\delta^{(1)}\delta^{(3)}\rangle + \langle\delta^{(3)}\delta^{(1)}\rangle + \langle\delta^{(2)}\delta^{(2)}\rangle] + ... \tag{107}$$

Translating this to the expression for the power spectrum, where we define $\langle\delta^{(i)}(\vec{q})\delta^{(j)}(\vec{k})\rangle = (2\pi)^3\delta^{(3)}(\vec{k}+\vec{q})P_{ij}(k)$, the full power spectrum, upto order ϵ^4 becomes the sum

$$P(k) = P_{11}(k) + 2P_{13}(k) + P_{22}(k) \tag{108}$$

Armed with this prescription and the Green's function, we can first write the second order solutions for δ by multiplying the Green's function with the source terms we had ignored earlier, ignoring the effective stress energy tensor terms (since they are third order). Higher order terms can be written as integrals using the Green's functions perturbatively. For instance, at second order, we have

$$\begin{aligned}\delta^{(2)}(\vec{k},a) = \frac{1}{16\pi^3 D^2(a_0)}\Bigg[&\left(\int_0^a dbG(a,b)b^2\mathcal{H}^2(b)D'^2(b)\right)\left(2\int d^3q\beta(\vec{q},\vec{k}-\vec{q})\delta s(\vec{k}-\vec{q})\delta s(\vec{q})\right)\\ &+\left(\int_0^a dbG(a,b)\left(2b^2\mathcal{H}^2(b)D'^2(b) + 3\mathcal{H}_0^2\Omega_m\frac{D^2(b)}{b}\right)\right)\\ &\times\left(\int d^3q\alpha(\vec{q},\vec{k}-\vec{q})\delta s(\vec{k}-\vec{q})\delta s(\vec{q})\right)\Bigg]\end{aligned} \tag{109}$$

This corresponds to a perturbative solution for the δ, as is common in field theory.

Finally, we can also treat the terms coming from the effective stress energy tensor as a third order source, and write down the solution due to those using the Green's function. First, note that the expression for $\theta^{(1)}(\vec{k},a)$ can be extracted from the solution to $\delta^{(1)}(\vec{k},a)$ using the continuity equation in Fourier space (98) at first order:

$$\theta^{(1)}(\vec{k},a) = -a\mathcal{H}\partial_a\delta^{(1)}(\vec{k},a) = -a\mathcal{H}\frac{D'(a)}{D(a)}\delta^{(1)}(\vec{k},a) \tag{110}$$

Now, the source term arising from the effective stress energy is $k^2c_s^2\delta - \mathcal{H}^{-1}k^2c_v^2\theta$, we can use the above expression to write the source only in terms of δ,

$$j(\vec{k},a) = k^2\Big[c_s^2 + a\frac{D'(a)}{D(a)}c_v^2\Big]\delta(\vec{k},a) \tag{111}$$

Then, then the third order solution due to this effective stress energy can be computed simply using the Green's function as

$$\delta_j^{(3)}(\vec{k},a) = -\frac{k^2}{D(a_0)}\int_0^a dbG(a,b)\Big[c_s^2 + a\frac{D'(a)}{D(a)}c_v^2\Big]D(b)\delta s(\vec{k}) \tag{112}$$

Now, since our power spectrum is up to fourth order in the perturbed δ, another fourth order term gets added to it through the coupling of $\delta^{(1)}$ and $\delta_j^{(3)}$, and gets added to the expression for the total power spectrum. This new term is defined in a similar way to the others:

$$\langle\delta^{(1)}(\vec{q})\delta_j^{(3)}(\vec{k})\rangle = (2\pi)^3\delta^{(3)}(\vec{k}+\vec{q})P_{13j}(k) \tag{113}$$

Since the higher order solutions are written in terms of $\delta^{(1)}$, the higher order power spectra can be isolated in terms of the first order power spectrum. We use Wick's theorem to compute the correlation function of the first order solutions that get isolated, which are treated as free fields.

Taking these Wick contractions, we can express the two-point correlation functions in terms of the first order solutions. Now, the first order solutions can also be expressed in terms of the *current* density contrast, using a multiplicative growth factor. Thus finally, the power spectrum corrections can be written in terms of the current time power spectrum.

Using this, one can generate corrections to the large-scale power spectrum due to small-scale structures numerically, *without having to resort to large simulations.* The current time power spectrum can be generated using common software like CAMB and then used as input to the expressions for the corrections. One only needs to run small, inexpensive simulations to generate the parameters that define the effective theory, and then the correct power spectra at large scales can be calculated.

The following are the expressions for the three corrective terms to the power spectrum at current time.

$$P_{22}(k,a_0) = \frac{k^3}{16\pi^2 D^4(a_0)} \int dq d(\cos\theta) \frac{1}{(k^2 - 2\cos\theta kq + q^2)^2} P_{11,l}(q) P_{11,l}(|\vec{k} - \vec{q}|)$$
$$\Big[\Big(\int_0^{a_0} dbb^2 G(a_0,b)\mathcal{H}^2(b)D'^2(b)\Big) 4\cos\theta(k - q\cos\theta)$$
$$+ 3\mathcal{H}_0^2\Omega_m\Big(\int_0^{a_0} dbG(a_0,b)\frac{D^2(b)}{b}\Big)(\cos\theta(k - 2q\cos\theta) + q)\Big] \times$$
$$\Big[\Big(\int_0^{a_0} dbb^2 G(a_0,b)\mathcal{H}^2(b)D'^2(b)\Big)(3k^2\cos\theta - kq(4\cos^2\theta + 1) + 2q^2\cos\theta)$$
$$+ 3\mathcal{H}_0^2\Omega_m\Big(\int_0^{a_0} dbG(a_0,b)\frac{D^2(b)}{b}\Big)(k^2 - 2kq\cos\theta + q^2)\Big] \quad (114)$$

Where $\cos\theta = \hat{k}\cdot\hat{q}$.

$$P_{13}(k,a_0) = -\frac{2k^3}{96(2\pi)^2 D^3(a_0)} P_{11,l}(k)$$
$$\int_0^\infty \frac{dr}{r^3}\Big[12r^7\mathcal{D}_4 - 24r\mathcal{D}_5 + 4r^3(16\mathcal{D}_1 + 8\mathcal{D}_2 + 4\mathcal{D}_3 - 3\mathcal{D}_4 + 24\mathcal{D}_5)$$
$$+ 8r^5(4\mathcal{D}_2 + 2\mathcal{D}_3 - 6\mathcal{D}_4 + 3\mathcal{D}_5 - 4\mathcal{D}_6)$$
$$+ 6(r^2-1)^3(r^2\mathcal{D}_4 + 2\mathcal{D}_5)\log\Big(\frac{1-r}{1+r}\Big)\Big] P_{11,l}(kr) \quad (115)$$

Where the expressions for $\mathcal{D}_{1\cdots 6}$ are supplied in Appendix B of [2].

$$P_{13j}(k,a_0) = -2\frac{k^2}{D(a_0)}\int_0^{a_0} dbG(a_0,b)\Big[c_s^2 + a\frac{D'(a)}{D(a)}c_v^2\Big] D(b)P_{11,l}(k) \quad (116)$$

6 Future Work and Summary

Currently, we are in the process of writing numerical codes to solve the second-order differential equation for the Green's function that propagates the solution $\delta(k)$ for any general case.

The user should be able to input the background cosmology by specifying the exact functional form of the Hubble parameter in terms of the scale factor a. The code then assumes a growth factor $D(a) \propto a$, which is hardwired into it to mimic a matter-dominated universe. This can be changed to any other form in the code. Using these two ingredients, and the background cosmological parameters, the code then solves for the Green's function that satisfies equation 101.

The code sets up a two dimensional grid using axes in a and b on which the Green's function $G(a, b)$ is constructed. For each value of b, the differential equation is solved for all a using an ODE solver in-built into the SCIPY package of the Python language. The boundary conditions given by (102) are then matched to the array of solutions for each b. This strategy is adopted because at each b, a different set of boundary conditions are imposed.

This set of solutions for different values b, constructed using the boundary conditions, is then stacked in a two dimensional array. Since this is a retarded Green's function, the below-diagonal terms of the array are all set to zero by hand, before the stacking is done. So, for each b, the solutions array starts from $a = b$ and goes up to $a = 1$, which denotes the current time. An array of zero values is stacked to the left of this array of solutions and then this becomes the complete solution for all a, where the value is $G = 0$ for $a < b$.

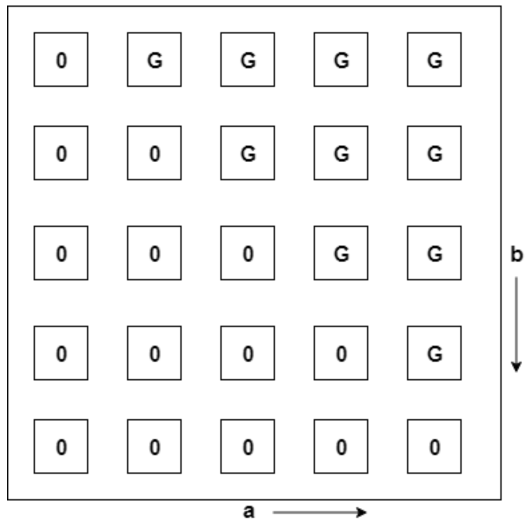
The structure of the numerical array is shown in figure 1. A similar array is constructed for the function $\partial_a G(a, b)$.

This stacking with zeros also gives the array a form which naturally imposes limits on the source term. The power spectrum corrections (114 - 116) include various time integrals of the form

$$\int_0^a G(a, b) S(b) db \tag{117}$$

Where $S(b)$ is a source. The structure of the arrays ensure that a numerical integration that goes vertically (i.e. integration over the variable b) naturally has zero values after the point $b = a$ for each value of a. This is in keeping with the structure of a retarded Green's function, and the limit that the source has to be active **before** the current time of the impact of the source, is automatically imposed.

Figure 1: The structure of the Green's Function array.



Using these arrays and the expressions for $D(a)$ and $\mathcal{H}(a)$, we can compute all the integrals that constitute the expressions for the correction to the power spectra (114 - 116). We are in the process of writing codes for all the power spectrum corrections. The final power spectrum is

$$P(k) = P_{11}(k) + P_{22}(k) + P_{31}(k) + P_{13j}(k) \tag{118}$$

The advantage of this approach is that all the corrections are expressed directly in terms of the linear theory matter power spectrum, which can be easily computed using fitting functions for a wide variety of cosmological models, and also by using publicly available codes. For this project, we seek to use the public code CAMB (Code for Anisotropies in the Microwave Background) to produce the linear power spectrum, and integrate its usage into our own code.

Then, given the input of the various fluid parameters, the background cosmology, and the linear power spectrum, the user can calculate, through this code all corrections arising in the power spectrum due to small-scale inhomogeneities.

A possible extension is to run small N-body simulations to extract the fluid parameters as one wishes, and then use this code to compute corrections in the power spectrum.

Summary The effective field theory approach, when applied to large-scale cosmological structures, yields a simple yet powerful tool to quantify structures on the largest scales while also accounting for the changes in structure formation due to small-scale inhomogeneities. It uses a perturbative approach to build higher order solutions based on an effective fluid generated by small wavelength matter density fluctuations, and parameterized by a small set of numbers which can be easily computed from inexpensive N-body simulations, or using existing simulation data. Our code can then be used to precisely and quickly calculate all corrections to the power spectrum given any cosmological model. This is effective in analytically describing large-scale structure without resorting to simulations of large size and fine resolution, both of which demand computational resources.

References

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