USING PROPER TIME

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Mapping Trajectories

In nonrelativistic classical mechanics, the program is to find the path of the particle in time, i.e. the curves $\mathbf{x}(t)$. Here, t is an invariant parameter that is equivalent to the 'clock' moving with the moving particle, which also measures the same time as the clock in the lab frame. The elimination of the parameter of time then enables us to plot the trajectory of the particle.

In relativistic mechanics, however, the coordinate t itself is changing from frame to frame. Because the fundamental nature of the Lorentz group mixes the time and space components, space and time are treated on an equal footing, and *both* become coordinates that change along the path and transform between frames. Thus, now we must find a new method to parameterise these **two variables,** x **and** t in terms of some other invariant parameter.

Another way to look at this is, now we have t as another coordinate, along with the space coordinate, and the *trajectory of the particle in spacetime (not space)* is now given by a plot of x vs t (for the case of one space dimension). Contrast this with normal space. The trajectory is given by an x vs y plot, and x and y are separate parameterised curves of the parameter t: $(x(t), y(t))$. Similarly, for a relativistic particle, the full trajectory is given by a graph of x vs t, because time is just another coordinate like space, and the full path requires the specification of **all the coordinates in spacetime**. Thus, it is only natural to model that both x and t are functions of another parameter denoted by τ.

We shall proceed to show how this parameter is the proper time. As is the case with any curve, one of the most elegant ways to parameterise it is by its **arc length**. The concept of proper time essentially does this. The proper time can be physically seen as a clock that moves with the moving particle. The expression is

$$d\tau^2 = dt^2 - dx^2$$

Which can clearly be seen as proportional to the infinitesimal line element in Minkowski spacetime. Thus, *the proper time is equivalent to the arc length of the world line of a moving particle.*

Notice that the four-velocity *is* defined as a derivative with respect to the proper time, and therefore must treat the spacetime coordinates as functions of proper time as well. This prescription is valuable as it obtains for us the spacetime coordinates as a function of this number which essentially measures motion along the world line. **The conventional definition of the four-velocity and four-acceleration then arise naturally from this parameterisation**. So, finally, our objective is: given the equations of motion of a particle in the space dimensions, obtain the parametrised curves $\mathbf{x}^\mu(\tau) = (x(\tau), t(\tau))$. The four-velocity definition then naturally becomes:

$$u^\mu = \frac{dx^\mu(\tau)}{d\tau}$$

Example 1

The simplest example we shall consider is that of a particle moving at the constant speed v along the x-axis, which starts from rest at $t = 0$. Simply written, the equation of motion of the particle is $x = vt$. We shall proceed to write x and t in terms of the proper time (arc length). The proper time is expressed as:

$$\tau = \int d\tau = \int \sqrt{dt^2 - dx^2}$$

We know that $vdt = dx$. Hence,

$$\tau = \int d\tau = dt \int \sqrt{1 - v^2}$$

or

$$\tau = t\sqrt{1-v^2}$$

We can easily invert this at find $t(\tau)$:

$$t(\tau) = \frac{\tau}{\sqrt{1-v^2}}$$

And, as $x = vt$, we have:

$$x(\tau) = \frac{v\tau}{\sqrt{1-v^2}}$$

The four-velocity comes out simply as:

$$u^\mu = \left(\frac{1}{\sqrt{1-v^2}}, \frac{v}{\sqrt{1-v^2}}\right)$$

Example 2

Consider a particle moving in one dimension with the velocity given by

$$\frac{dx}{dt} = \frac{gt}{\sqrt{1+g^2t^2}}$$

Again, the proper time is given by the expression:

$$\tau = \int_0^t dt\sqrt{1-\left(\frac{dx}{dt}\right)^2} = \int_0^t \frac{dt}{\sqrt{1+g^2t^2}} = \frac{1}{g}\sinh^{-1}(gt)$$

Hence

$$t(\tau) = \frac{1}{g}\sinh(g\tau)$$

Then,

$$x = \int_0^t v(t)dt = \int_0^t \frac{gtdt}{\sqrt{1+g^2t^2}} = \frac{1}{g}\sqrt{1+g^2t^2}$$

From the definition of $t(\tau)$, we have

$$x(\tau) = \frac{1}{g}\cosh(g\tau)$$

We can now easily compute the four-velocity of this particle.

The Relativistic Action

The expression for the proper time is

$$d\tau = ds = \sqrt{dt^2 - dx^2}$$

Now, consider two fixed events in spacetime that occur at the same point. The vertical straight line joining these two events is the path with the **maximum** spacetime interval. Any other path (that is not completely vertical) will pick up contributions from a change in the space coordinate, that get *subtracted* due to the signature of the Minkowski metric.

Now consider any two points in spacetime. We can draw a straight line between these two points, which represents the world line of a particle moving at some constant velocity. Now, we can easily transform this to a frame where the two events occur at the same place, so that this straight line becomes vertical, like described in the previous paragraph. Since the spacetime interval is invariant under transformations, this straight line, in any inertial frame, will have **the maximum spacetime interval**.

Thus, any straight line between two spacetime events, which is also the world line of a **free particle** (moving at a constant speed), is the line that **maximises the spacetime interval**, and thus is the curve of **maximum proper time**. This sounds like a variational principle, and using it we shall derive the equation for a free particle.

Let the action be denoted by the proper time (σ is a rescaled parameter describing the path, proportional to the proper time along the path):

$$\tau = \int_A^B dt \left[\left(1 - \left(\frac{dx}{dt} \right)^2 \right] \right]^{1/2}$$

The negative of this has to be minimised. We can read off this the Lagrangian (upto a multiplicative constant):

$$L = -\sqrt{1 - v^2}^{1/2}$$

Now, using the Euler-Lagrange equations on this Lagrangian, we have:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial u^\alpha} \right) = \frac{\partial L}{\partial x^\alpha}$$

Which gives

$$\frac{d}{dt}\left(\frac{1}{L}\frac{dx^\alpha}{dt}\right) = 0$$

As $L = -\sqrt{1-v^2}$, this implies,

$$\frac{d}{dt}\left(\sqrt{1-v^2}\frac{dx^\alpha}{dt}\right) = 0$$

We can replace the factor $dt/\sqrt{1-v^2}$ by $d\tau$, and the time derivative with the proper time derivative (it picks up only a multiplicative factor), and thus write:

$$\frac{d}{d\tau}\left(\frac{dx^\alpha}{d\tau}\right) = 0$$

Or

$$\frac{d^2x^\alpha}{d\tau^2} = 0$$

Which is exactly the equation for a free particle!

We also know, from the fundamental definition of momentum, that

$$p^\alpha = \partial L/\partial u^\alpha$$

Taking our given form of the Lagrangian, we can easily see how

$$p^\alpha = \frac{\partial L}{\partial u^\alpha} = -\frac{1}{L}\frac{dx^\alpha}{dt} = -\frac{dx^\alpha}{d\tau}$$

From this, we can also recover the multiplicative constant that we had left earlier. It turns out to be $-m$, which makes the momentum

$$p^\alpha = m\frac{dx^\alpha}{d\tau}$$

Putting in the expression for dt in terms of $d\tau$, we recover the more common expression for the relativistic momentum:

$$p^i = \frac{mv^i}{\sqrt{1-v^2}}$$

where v^i denotes the components of the three-velocity (in our case only one component).

We can now also use the Hamiltonian formalism to directly write an expression for the total energy of the particle. The full Lagrangian is now $L = -m\sqrt{1-v^2}$. The Hamiltonian can be written as:

$$H = p \cdot v - L = m\frac{dx}{d\tau}\frac{dx}{dt} + m\sqrt{1-v^2} = \frac{mv^2}{\sqrt{1-v^2}} + m\sqrt{1-v^2} = \frac{m}{\sqrt{1-v^2}}$$

In fact, now we can take the component $\alpha = 0$ in our earlier definition of momentum and **automatically identify** that the time-component of this four-vector is the total energy:

$$p^0 = m\frac{dt}{d\tau} = \frac{m}{\sqrt{1-v^2}}$$

Thus, we can see that we have redefined energy and momentum using the fundamentals of mechanics using the proper time as an analogue of the action that is extremised along the curve followed by a particle.
